Fundamental theorem of homomorphism of groups.

Theorem → State and prove the fundamental theorem of homomorphism of groups.

Proof: — Let f be a homomorphism of G onto G' then by definition G' is the homomorphic image of a group G.

Let K be the kernel of this homomorphism f. Then K is a normal subgroup of G and so G/K is a Quotient group.

Here we have to show that $G/K \cong G'$ i.e. G/K is isomorphic to homomorphic image G'

Let $a \in G$ then $f(a) \in G'$ $[\because f: G \to G']$

Also if $a \in G$, then $Ka \in G/K$ since G/K is the set of all right (or left) cosets of K in G. Let us construct a mapping $\phi: G/K \to G'$ such that $\phi(Ka) = f(a) \forall a \in G$.

We shall show that ϕ is an isomorphism of G/K onto G'

(i) ϕ is one-one:

We have $\phi(Ka) = \phi(Kb)$

$$\Rightarrow f(a) = f(b)$$

$$\Rightarrow f(a)[f(b)]^{-1} = f(b)[f(b)]^{-1}$$

$$\Rightarrow f(a) f(b^{-1}) = f(b) f(b^{-1})$$

$$\Rightarrow f(a) f(b^{-1}) = e'$$

$$\Rightarrow f(ab^{-1}) = e' \quad [\because f \text{ is homomorphism}]$$

$$\Rightarrow Ka = Kb \in K \quad [\because K \text{ is kernel}]$$

$\therefore \phi$ is one-one.

let $y \in G'$ then $y = f(a)$ for some $a \in G$,
Since f is onto G'.

Also $Ka \in G/K$ Then $\phi(Ka) = f(a) = y$

$\therefore \phi$ is onto map.

(iii) ϕ preserves operations:

We have $\phi[(Ka)(Kb)] = \phi(Kab) = f(ab)$

$= f(a) f(b)$ [$\because f$ is homomorphism]

$= \phi(Ka)\,\phi(Kb)$

Hence ϕ is an isomorphism of G/K onto G'

$\therefore G/K \cong G'$.

Q→ Prove that every isomorphic image of a cyclic group is again cyclic.

Solution:— let $G = \{a\}$ be cyclic group generated by a.

let G' be an isomorphic image of G under the isomorphism f i.e. $f: G \to G'$

let $f(a^n) \in G'$ be the image of $a^n \in G$.

We have

$f(a^n) = f(aaa \text{ --- to } n \text{ factors})$

$= f(a) f(a) f(a) \text{ --- to } n \text{ factors}$

[$\because f$ is an isomorphism]

$= [f(a)]^n$.

i.e. every element of G' can be expressed as an integral power of $f(a)$.

Thus G' is cyclic and $f(a)$ is a generator of G'.

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